Expander Graphs
Exercise Sheet 6

Question 1. Show that if $E\colon \{0,1\}^n \times \{0,1\}^d \to \{0,1\}^m$ is a $(k_{\max}, \epsilon)$-lossless conductor then the associated bipartite graph $G = (L, R, E)$ is a $(K_{\max}, \epsilon)$-lossless expander where $K_{\max} = 2^{k_{\max}}$.

Question 2. Let H be a d-regular bipartite graph with partition classes of size s and let G be an s-regular bipartite graph with partition classes of size n. Show that the vertex expansion of $G \circledz H$ is at most d.

Question 3. Show that if Y is ϵ-close to uniform, X is a k-source, and E is a $(k_{\max}, a, \epsilon)$-conductor with $k \leq k_{\max}$, then $E(X,Y)$ is a $(k + a, 2\epsilon)$-source.

Question 4. Prove one of the following:

1. A (k, ϵ)-lossless conductor $E\colon \{0,1\}^n \times \{0,1\}^d \to \{0,1\}^m$ exists for all $m > k + d + \log \frac{1}{\epsilon}$.
2. An (a, ϵ)-extracting conductor $E\colon \{0,1\}^n \times \{0,1\}^d \to \{0,1\}^m$ exists for all $d > a + 2\log \frac{1}{\epsilon}$.
3. An (a, ϵ)-buffer conductor $E\colon \{0,1\}^n \times \{0,1\}^d \to \{0,1\}^m \times \{0,1\}^b$ exists for all $d > a + 2\log \frac{1}{\epsilon}$ and $m + b > n + d + \log \frac{1}{\epsilon}$.

(Don't worry about achieving the optimal conditions on the parameters)

Question 5. Suppose $\epsilon > 0$ and $\boldsymbol{p}$ is a distribution with $||\boldsymbol{p}||_2^2 \leq 2^{-a}$. Show that $\boldsymbol{p}$ is ϵ-close to a distribution $\boldsymbol{q}$ with $H_\infty(\boldsymbol{q}) \geq b - \log \frac{1}{\epsilon}$.

Let G be an (N, D, α)-graph where $N = 2^n$ and $D = 2^d$ and let $\langle E, C \rangle : \{0,1\}^n \times \{0,1\}^d \to \{0,1\}^n \times \{0,1\}^d$ be a *rotation map* for G. Show that, for $\epsilon > 0$, E is a $\left(n - d, 2\log\left(\frac{1}{\alpha}\right) - \log\frac{1}{\epsilon} - 1, \epsilon\right)$ conductor.

(Hint: Note that $E(\boldsymbol{p}, \boldsymbol{u}) = \hat{A}\boldsymbol{p}^T$. Try to bound its 2-norm)